Vector Analysis Examples

EX-1 If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$,

prove that $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$.

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1\mathbf{i}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_2\mathbf{j}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_3\mathbf{k}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1B_1\mathbf{i}\cdot\mathbf{i} + A_1B_2\mathbf{i}\cdot\mathbf{j} + A_1B_3\mathbf{i}\cdot\mathbf{k} + A_2B_1\mathbf{j}\cdot\mathbf{i} + A_2B_2\mathbf{j}\cdot\mathbf{j} + A_2B_3\mathbf{j}\cdot\mathbf{k} + \\
&\quad A_3B_1\mathbf{k}\cdot\mathbf{i} + A_3B_2\mathbf{k}\cdot\mathbf{j} + A_3B_3\mathbf{k}\cdot\mathbf{k} \\
&= A_1B_1 + A_2B_2 + A_3B_3
\end{aligned}$$

since $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$ and all other dot products are zero.

ملاحظة : طباعة الحرف بالشكل الغامق يعني ذلك انه متجه

ملاحظة : في بعض المصادر تستخدم الرموز :

$A_z = A_3$ ، $A_y = A_2$ ، $A_x = A_1$ $\quad$ $k = a_z$ ، $j = a_y$ ، $i = a_x$

EX-2 Find the angle between $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \quad A = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3, \quad B = \sqrt{(6)^2 + (-3)^2 + (2)^2} = 7$$

$$\mathbf{A}\cdot\mathbf{B} = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4$$

Then $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \dfrac{4}{(3)(7)} = \dfrac{4}{21} = 0.1905$ and $\theta = 79^\circ$ approximately.

EX-3 If $\mathbf{A}\cdot\mathbf{B} = 0$ and if A and B are not zero, show that $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

If $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = 0$, then $\cos\theta = 0$ or $\theta = 90^\circ$.

Conversely, if $\theta = 90^\circ$, $\mathbf{A}\cdot\mathbf{B} = 0$.

EX-4 If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, show that $A = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$.

$\mathbf{A}\cdot\mathbf{A} = (A)(A)\cos 0^\circ = A^2$. Then $A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$.

Also,
$$\mathbf{A}\cdot\mathbf{A} = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$
$$= (A_1)(A_1) + (A_2)(A_2) + (A_3)(A_3) = A_1^2 + A_2^2 + A_3^2$$

EX-5 If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and if $\mathbf{A}$ and $\mathbf{B}$ are not zero, show that $\mathbf{A}$ is parallel to $\mathbf{B}$.

If $\mathbf{A} \times \mathbf{B} = AB \sin\theta \, \mathbf{u} = \mathbf{0}$, then $\sin\theta = 0$ and $\theta = 0^\circ$ or 180°.

EX-6 Show that $|\mathbf{A} \times \mathbf{B}|^2 + |\mathbf{A} \cdot \mathbf{B}|^2 = |\mathbf{A}|^2 |\mathbf{B}|^2$.

$$
\begin{aligned}
|\mathbf{A}\times\mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = |AB \sin\theta \, \mathbf{u}|^2 + |AB \cos\theta|^2 &= A^2B^2 \sin^2\theta + A^2B^2 \cos^2\theta \\
&= A^2B^2 = |\mathbf{A}|^2 |\mathbf{B}|^2
\end{aligned}
$$

EX-7 If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, prove that $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$.

$$
\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \times (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1\mathbf{i} \times (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_2\mathbf{j} \times (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_3\mathbf{k} \times (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1B_1\mathbf{i}\times\mathbf{i} + A_1B_2\mathbf{i}\times\mathbf{j} + A_1B_3\mathbf{i}\times\mathbf{k} + A_2B_1\mathbf{j}\times\mathbf{i} + A_2B_2\mathbf{j}\times\mathbf{j} + A_2B_3\mathbf{j}\times\mathbf{k} + A_3B_1\mathbf{k}\times\mathbf{i} + A_3B_2\mathbf{k}\times\mathbf{j} + A_3B_3\mathbf{k}\times\mathbf{k} \\
&= (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.
\end{aligned}
$$

EX-8 If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, find (*a*) $\mathbf{A} \times \mathbf{B}$, (*b*) $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B})$.

$$(a)\quad \mathbf{A}\times\mathbf{B} = (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$$

Another Method.

$$(2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - 3\mathbf{j} \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - \mathbf{k} \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$= 2\mathbf{i}\times\mathbf{i} + 8\mathbf{i}\times\mathbf{j} - 4\mathbf{i}\times\mathbf{k} - 3\mathbf{j}\times\mathbf{i} - 12\mathbf{j}\times\mathbf{j} + 6\mathbf{j}\times\mathbf{k} - \mathbf{k}\times\mathbf{i} - 4\mathbf{k}\times\mathbf{j} + 2\mathbf{k}\times\mathbf{k}$$

$$= \mathbf{0} + 8\mathbf{k} + 4\mathbf{j} + 3\mathbf{k} - \mathbf{0} + 6\mathbf{i} - \mathbf{j} + 4\mathbf{i} + \mathbf{0} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$$

$$(b)\quad \mathbf{A} + \mathbf{B} = (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) + (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$\mathbf{A} - \mathbf{B} = (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) - (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = \mathbf{i} - 7\mathbf{j} + \mathbf{k}$$

Then $$(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B}) = (3\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (\mathbf{i} - 7\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 1 & -7 \end{vmatrix} = -20\mathbf{i} - 6\mathbf{j} - 22\mathbf{k}.$$

Another Method.

$$(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B}) = \mathbf{A} \times (\mathbf{A} - \mathbf{B}) + \mathbf{B} \times (\mathbf{A} - \mathbf{B})$$

$$= \mathbf{A}\times\mathbf{A} - \mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{A} - \mathbf{B}\times\mathbf{B} = \mathbf{0} - \mathbf{A}\times\mathbf{B} - \mathbf{A}\times\mathbf{B} - \mathbf{0} = -2\mathbf{A}\times\mathbf{B}$$

$$= -2(10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}) = -20\mathbf{i} - 6\mathbf{j} - 22\mathbf{k}, \quad \text{using } (a).$$

THE GRADIENT

EX-9 If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\,\mathbf{k}$$

$$= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k}$$

$$= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$$

EX-10 Find $\nabla\phi$ if (a) $\phi = \ln|\mathbf{r}|$, (b) $\phi = \frac{1}{r}$.

(a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|\mathbf{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$.

$$\begin{aligned}
\nabla\phi &= \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2) \\
&= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\} \\
&= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}
\end{aligned}$$

(b)
$$\begin{aligned}
\nabla\phi &= \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left\{(x^2+y^2+z^2)^{-1/2}\right\} \\
&= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} \\
&= \mathbf{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \mathbf{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \mathbf{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\} \\
&= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}
\end{aligned}$$

THE DIVERGENCE

EX-11 If $\mathbf{A} = x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla\cdot\mathbf{A}$ (or div $\mathbf{A}$) at the point $(1,-1,1)$.

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \quad \text{at } (1,-1,1).$$

EX-12 Given $\phi = 2x^3y^2z^4$. Find $\nabla\cdot\nabla\phi$ (or div grad ϕ).

$$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k}$$

Then $$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Laplacian operator $\equiv \nabla^2$

EX-13 Show that $\nabla\cdot\nabla\phi = \nabla^2\phi$, where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

denotes the Laplacian operator

$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

EX-14 Prove that $\nabla^2(\frac{1}{r}) = 0$.

$$\nabla^2(\frac{1}{r}) = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(\frac{1}{\sqrt{x^2+y^2+z^2}})$$

$$\frac{\partial}{\partial x}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{\partial}{\partial x}[-x(x^2+y^2+z^2)^{-3/2}]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}(\frac{1}{\sqrt{x^2+y^2+z^2}}) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(\frac{1}{\sqrt{x^2+y^2+z^2}}) = 0.$$

The equation $\nabla^2\phi = 0$ is called *Laplace's equation.*

THE CURL

EX-15 If $\mathbf{A} = xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k}$, find $\nabla \times \mathbf{A}$ (or curl $\mathbf{A}$) at the point $(1,-1,1)$.

$$\nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\mathbf{k}$$

$$= (2z^4 + 2x^2y)\,\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

EX-16 If $\mathbf{A} = x^2y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find curl curl $\mathbf{A}$.

$$\text{curl curl }\mathbf{A} = \nabla\times(\nabla\times\mathbf{A})$$

$$= \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \nabla\times\left[(2x+2z)\mathbf{i} - (x^2+2z)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+2z & 0 & -x^2-2z \end{vmatrix} = (2x+2)\mathbf{j}$$